SECTION 1.3 : $|\Omega| = \infty$

1.16 Toss a coin until the first tails shows up. What is your space of outcomes?

An outcome is a sequence

Ex : (H,T) 2nd toss is tails

(T) 1st toss i tails

$(\underbrace{H, \dots}_{k-1 \text{ heads}} \overset{k^{th}\text{ toss}}{H,T})$ k^{th} toss is tails.

$\Omega_1 = \{(T), (H,T), (H,H,T), \dots\}$. → space of weird tuples

This is in bijection with the following set

th toss at which tails fis shows up

$\Omega_2 = \{1, 2, 3, \dots \} \cup \{\infty\}?$

k represents the toss at which you get tails.
What if tails never shows up?

POLL :

Is ∞ included in Ω_2 ?

YES | NO.

"apriori"

So write

$\Omega = \{1, 2, 3, \ldots, \infty\}$

∞ is just some symbol that represents tails never showing up.

Now time to figure out probabilities.

$P(\{1\}) = P(\text{First toss is tails}) = \frac{1}{2}$

$P(\{2\}) = P(\{(H,T)\}) = \frac{1}{4} = \frac{1}{2^2}$

(since there are 4 possibilities (HH) (HT) (TH) (TT)).

$P(\{k\}) = P(k) \overset{?}{=}$

In general, what is

$P(\{k\})$? Here $\tilde{\Omega}$ consists of k tuples

cartesian product

$\Rightarrow \tilde{\Omega}_k = \{H,T\} \times \{H,T\} \times \cdots \times \{H,T\}$

$\Rightarrow |\tilde{\Omega}_k| = 2^k$

$P(\{k\}) = P(\{(H, H, H, H, \ldots, T)\}) = \frac{1}{2^k}$

$$= \frac{|\{(H, H, \dots, H, T)\}|}{|\tilde{\Omega}_k|} = \frac{1}{2^k}$$

This is because every sequence in $|\tilde{\Omega}_k|$ is equally likely.

Question: What is $P(\infty)$?

One can think of this as

$\lim_{n\to\infty} x_n = x_\infty$

$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$

↳ f is continuous

$$\lim_{k\to\infty} P(k) \overset{?}{=} P(\lim_{k\to\infty} k)$$

"∞"

"measure"

$$= \lim_{k\to\infty} \frac{1}{2^k} = 0$$

POLL

Do you think that these manipulations were justified?

YES | NO

Let us do this another way and use the rules of probability. Then

what kind of union?

$$P(\Omega) = P(\{\infty\} \sqcup \{1\} \cup \{2\} \cdots)$$

rules of prob $\Rightarrow$ add probabilities

$$= P(\infty) + P(1) + P(2) + \qquad + P(k) + \cdots$$

$$= P(\infty) + \underbrace{\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots}_{\text{What is this object?}}$$

∞ series.

$\downarrow$

Geometric series.

$$= P(\infty) + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} \qquad — \quad (1)$$

$$=: \sum_{i=1}^{\infty} \frac{1}{2^i}$$

GEOMETRIC SERIES

Let $S_n(x) = 1 + x + x^2 + \cdots + x^n$ $\qquad x = \frac{1}{2}$

$$= \sum_{i=0}^{n} x^i$$

what is this term? $x^0 = 1$

Let $S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} x^i =: \sum_{i=0}^{\infty} x^i$

$$S_n(x) = 1 + x + x^2 + \cdots \quad x^n$$

$$- x S_n(x) = -(x + x^2 + x^3 + \cdots + x^{n+1})$$

$\Rightarrow \quad S_n(x) - x S_n(x) = 1 - x^{n+1} \quad \Rightarrow \quad (1-x) S_n(x) = 1 - x^{n+1}$

$S_n = \dfrac{1 - x^{n+1}}{1 - x} \qquad \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \dfrac{1 - x^{n+1}}{1 - x} =: S(x)$

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & |x| < 1 \\ 1 & x = 1 \\ \infty & x > 0,\ |x| > 1 \\ \text{undef.} & x < 0,\ |x| \leq 0 \end{cases} \quad \Rightarrow \quad S(x) = \frac{1}{1-x}$$

Recall (1) $= \displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} S\left(\frac{1}{2}\right)$

OR $S\left(\frac{1}{2}\right) - 1$

$i = 0$ term

$$= \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2} \cdot \frac{2}{1} = 1$$

$$\Rightarrow \quad P(\infty) + \underbrace{S\left(\tfrac{1}{2}\right)}_{} = 1 \quad \Rightarrow \quad P(\infty) = 0$$

($S\left(\frac{1}{2}\right) = 1$)

So far we have talked about one kind of infinite

$\Omega = \{1, 2, 3, \ldots \}$

Here is another kind of ∞ sample space.

1.17 Uniform distribution.

→ interval of real # between 0 & 1.

$\Omega = [0,1]$.

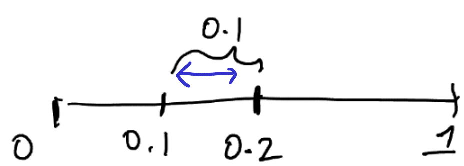
Example of an outcome $X = 0.372$

$P(X = 0.372) = 0$

maybe. let X be this ~~outcome~~.

$\mathbb{P}(X \text{ in } (0.1, 0.2)) \stackrel{?}{=} 0.1$

$= 0.2 - 0.1$



= proportional to interval length.

Georg Cantor } early 1900s
↳ wikipedia
Cantor's diagonal argument.

POLL

What is $\mathbb{P}(X = 0.4)$?

This is fine! All of this makes sense

$|[0,1]| \text{"="} \infty$

uncountable $> |\{1, 2, \ldots\}|$ countable

A	B	C
$1/2$	0.4	0 ✓

$$\mathbb{P}(X = 0.4) = \lim_{\epsilon \downarrow 0} \mathbb{P}\big(X \in (0.4-\epsilon, 0.4+\epsilon)\big)$$

$=$

$(a,b) \subset [0,1]$ Then $P(A) = \dfrac{|A|}{|[0,1]|} = \dfrac{b-a}{1}$ "length"

In general if $(a,b) \subset (c,d)$ $P((a,b)) = \dfrac{b-a}{d-c}$

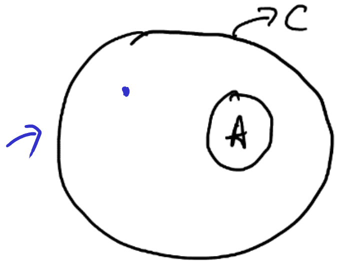
UNIFORMLY DISTRIBUTED POINTS ON THE PLANE

2d-plane

Pick ~~poi~~ point uniformly in the disc.

1-17

C (for circle)



Throw a dart.

at the board.

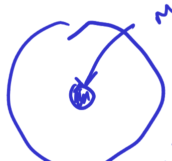
What is the prob that the dart falls in the set A?

Assume all points are equally likely.

$$\mathbb{P}(A) = \frac{\text{"Area" of } A}{\text{Area of Circle.}}$$

"Generalized" length

Notice the length is replaced by area.



more likely to go for the center in baby darts.

Is this a good model for the game of darts?

POLL

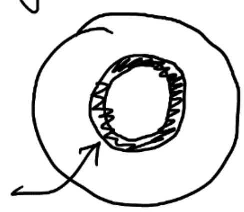
A

YES

B

NO

In a real game of darts



So people aren't really throwing it uniformly on the dartboard.

What we have done so far.

1) Learned to count, when to multiply.

2) Rules of probability

3) Equally likely outcomes in finite and ∞ systems.